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Solitary Waves Interacting with an External Field

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It is shown that the equations of a solitary wave interacting with an external field can be obtained from the noninteraction equations and a substitution analogous to the prescription of quantum mechanics for the energy and momentum operators in the presence of an interaction. Next it is shown that, if the rate of change of the external field is sufficiently small, then the motion of the solitary wave as a whole is identical to that of a point charge in an electromagnetic field or to that of a point mass in a given interaction potential. This identity holds regardless of the specific solitary wave equation. An estimate for the external field maximal rate of change is derived.

1. INTRODUCTION

The term *solitary wave* will appear many times in the following discussion. Since there seems to be no universally accepted definition for it, the meaning assigned to it, at least for the purposes of this paper, will be stated in the following:

SW Definition. A solitary wave (SW) is a solution of any one partial differential equation or a system of such equations in three or less independent space variables and the time, which is derivable from a Lagrangian density, such that the integral over all space of that density is finite and nonzero for all time when a SW solution is substituted in it.

Stated briefly, a solitary wave is a solution of the above class of differential equations which is localized in space for all time. A wave packet, for example, is not a solitary wave.

The most-studied SW solutions are those of the nonlinear Klein–Gordon (KG) equation. The first to investigate the stability of such solutions were Anderson and Derrick (1970; Anderson, 1971). Later work treated the ques-

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tion as a problem in functional analysis (Berger, 1972; Grillakis *et al.*, 1987; Strauss, 1989) and obtained stability criteria related to the nonlinear term in the KG equation. Lee (1981) included a short discussion on this subject in his book. The mutual interaction between two nonstable (finite lifetime) SWs of a real nonlinear KG equation was studied by Okolowski and Slomiana (1988), who showed that this interaction is due to the same nonlinearity which is responsible for their localization in space. A model of leptons was given by Cooperstock and Rosen (1989) based on a complex system of four interacting fields, three of which are scalar and one of which is a 4-vector field. Friedberg *et al.* (1976) reported on an extensive investigation of a system of two interacting scalar fields, represented by two coupled nonlinear KG equations, and intended to serve as a possible model of elementary particles.

Although there is no lack of publications related to solitary waves, I have not found any which treat the interaction of solitary waves with a given external field. However, there are several works on the Klein–Gordon–Maxwell system (Rosen, 1939; Deumens, 1986; Kobushkin and Chepilko, 1990) which share some common ground with this paper. The underlying idea is the same in all three. The solitary waves are formed from a scalar complex field interacting with an electromagnetic (EM) field. The last is considered to be the sum of external and internal components. The internal component is assumed to be created by the scalar field via the Maxwell equations. Hence, the last become nonlinearly coupled to the KG equation of the scalar field. In spite of its attractiveness, this idea has not produced the results hoped for. The similarity with our work is in the way (gauge invariant) the EM potentials enter the KG equation. The difference is that our work treats these potentials as strictly external and given (just as in quantum mechanics). This is a very important distinction. It is the reason for mentioning the above three works. Had we made the same assumption as theirs, our results could not have been obtained.

In what follows we consider the interaction of the solitary waves (not just of the nonlinear KG equation, but those of all known and not yet known complex nonlinear equations which admit such solutions) with a given external field. Attention will be focused on the *macroscopic* aspect of the interaction—that is, the motion of the solitary wave as a whole. Questions concerning the variations in size and in shape, or of the SW stability due to the interaction, will not be addressed. To do the first, the SW position will be identified with the position of a suitably defined *center of its charge*. And to do that, one needs to have a well-defined *charge density* associated with the solitary wave, which in turn requires that the SW equations be derivable from a *Lagrangian density*. This is a natural requirement since the solitary waves are expected to possess certain conserved quantities, like charge and mass, which are well defined and their conservation guaranteed when such density exists.

If the motion of a solitary wave is affected by a given external field U_μ, it must be because that field enters as a variable in its differential equation and hence in its Lagrangian density

$$\mathcal{L}[\psi, U] = \mathcal{L}(\psi^*_{\rho,\nu}, \psi_{\rho,\nu}, \psi^*_\rho, \psi_\rho, U_\mu)$$

where ψ_ρ, ψ^*_ρ are the components of the SW wave function and their complex conjugates. $\psi_{\rho,\mu} = \partial\psi_\rho/\partial x_\mu$ and $\psi^*_{\rho,\mu} = \partial\psi^*_\rho/\partial x_\mu$ are their derivatives with respect to the space and time coordinates. (All densities in this paper are denoted with capital script letters.) The number of the wave function components is left unspecified, so that ψ_ρ can be a scalar field when $\rho = 1$, a 2-spinor field when $\rho = 1, 2$, a 4-spinor field when $\rho = 1, 2, 3, 4$, or a 4-vector field when $\rho = 0, 1, 2, 3$. The field U_μ with which the solitary wave interacts is assumed to be a real 4-potential or a scalar potential field. There are no restrictions on it, except that its free-field equations must also be derivable from a *Lagrangian density*

$$\mathcal{N}[U] = \mathcal{N}(U_{\mu,\nu}, U_\mu)$$

where $U_{\mu,\nu} = \partial U_\mu/\partial x_\nu$ for $\mu, \nu = 0, 1, 2, 3$ or for $\nu = 1, 2, 3$, $\mu = 0$.

The main objective of this work is to show that, under some very general assumptions, the space integral of the solitary wave Lagrangian density

$$L = \int \mathcal{L}[\psi, U]\, d^3x$$

is identical with the *Lagrangian function of a point charge* moving in an electromagnetic field (4-potential) or with the *Lagrangian function of a point mass* moving in a scalar potential. Since the second is a special case of the first, U will be assumed to be a 4-potential field.

2. THE LAGRANGIAN DENSITY FOR AN INTERACTING SOLITARY WAVE

Since $\mathcal{N}[U]$ is the free U-field Lagrangian density (no sources), it cannot depend on ψ_ρ. The Lagrangian density for the system of the interacting ψ-field and U-field is

$$\mathcal{T} = \mathcal{N}[U] + \mathcal{L}[\psi, U] \tag{1}$$

Now, the role of U in $\mathcal{L}[\psi, U]$ must be made more specific than just that of a parameter. One observes that a physical field acts on entities which by themselves are sources of fields of the same type. Thus, not only must the field sources, if distributed, be densities obeying a conservation law, but so must *the entities on which a field acts be densities obeying a conservation law.* Hence, we must demand that the U-field acts on the ψ-field via a set of four conserved densities $\mathcal{D}_\mu$ associated with the ψ-field. The densities $\mathcal{D}_\mu$ always exist, provided that the following condition holds.

Condition 1. The Lagrangian density $\mathcal{L}[\psi, U] = \mathcal{L}(\psi^*_{\rho,\nu}, \psi_{\rho,\nu}, \psi^*_\rho, \psi_\rho, U_\mu)$ is invariant under the transformation with parameter ϵ

$$\psi'_\rho = e^{i\epsilon}\psi_\rho, \qquad \psi'^*_\rho = e^{-i\epsilon}\psi^*_\rho \tag{2}$$

Then they are given by (the summation convention is assumed for the entire paper)

$$\mathcal{D}_\mu = i\psi_\rho \frac{\partial \mathcal{L}}{\partial \psi_{\rho,\mu}} - i\psi^*_\rho \frac{\partial \mathcal{L}}{\partial \psi^*_{\rho,\mu}} \tag{3}$$

and their existence is a consequence of Noether's theorem. They are conserved,

$$\frac{\partial \mathcal{D}_\mu}{\partial x_\mu} = \frac{\partial \mathcal{D}_0}{\partial t} + \nabla\cdot\overline{\mathcal{D}} = 0 \qquad \text{with} \quad t = x_0$$

and thus qualify to be the sources of a physical field. $\mathcal{D}_0$ and $\overline{\mathcal{D}} = (\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ are the density and the current density vector of the sources (a barred letter will be used to denote any 3-vector). Some outstanding discussions on Noether's theorem and its applications can be found in Goldstein (1981, Section 12.7), Logan (1977, Sections 2.4 and 8.4), and Olver (1993, Sections 4.4 and 5.3). Having defined the total Lagrangian density $\mathcal{T}$ allows us to write down the Euler–Lagrange equations for the U-field

$$\frac{d}{dx_\nu}\left(\frac{\partial \mathcal{T}}{\partial U_{\mu,\nu}}\right) - \frac{\partial \mathcal{T}}{\partial U_\mu} = 0$$

Taking into account (1), we find that this becomes

$$\frac{d}{dx_\nu}\left(\frac{\partial \mathcal{N}}{\partial U_{\mu,\nu}}\right) - \frac{\partial \mathcal{N}}{\partial U_\mu} = \frac{\partial \mathcal{L}}{\partial U_\mu} \tag{4}$$

Without the right-side term, these would be the equations of a free U-field. With it, they describe a source-driven U-field, and the sources, of course, are represented by just that term. Now, the statement U_μ *acts on* ψ_ρ *via* $\mathcal{D}_\mu$ can be made mathematically precise.

Condition 2. The sources of the U-field must be proportional to the set of densities $\mathcal{D}_\mu$ associated with the ψ-field. That is,

$$\frac{\partial \mathcal{L}}{\partial U_\mu} = ig\left(\psi_\rho \frac{\partial \mathcal{L}}{\partial \psi_{\rho,\mu}} - \psi_\rho^* \frac{\partial \mathcal{L}}{\partial \psi_{\rho,\mu}^*}\right) \tag{5}$$

where g is the constant of proportionality, which may be different for different types of SW equations.

This is the second qualification which $\mathcal{L}[\psi, U]$ has to meet if it is to describe a solitary wave in interaction with the U-field. Obviously, Condition 2 was not mathematically derived. It was accepted on the grounds of general considerations, as briefly outlined above. The correctness or incorrectness of this choice can be shown only by the results which are derivable from it. First, Condition 2 will be used to make the functional dependence of $\mathcal{L}[\psi, U]$ on U as specific as possible, while leaving its dependence on ψ as general as possible.

Theorem 1. Any Lagrangian density $\mathcal{L}[\psi, U]$ for which the set of densities $\mathcal{D}_\mu$ are defined by Condition 1 and which satisfies Condition 2,

$$\frac{\partial}{\partial U_\mu} \mathcal{L}[\psi, U] = g\mathcal{D}_\mu \tag{6}$$

is of the form

$$\mathcal{L}[\psi, U] = \mathcal{L}(\psi_{\rho,\mu}^* - igU_\mu\psi_\rho^*, \psi_{\rho,\mu} + igU_\mu\psi_\rho, \psi_\rho^*, \psi_\rho) \tag{7}$$

Proof. To see that any Lagrangian density of the above form satisfies Condition 2 it is sufficient to differentiate it with respect to U_μ. Next we observe that the equation

$$\frac{\partial \mathcal{L}}{\partial U_\mu} = ig\psi_\rho \frac{\partial \mathcal{L}}{\partial \eta_{\rho\mu}} - ig\psi_\rho^* \frac{\partial \mathcal{L}}{\partial \eta_{\rho\mu}^*}$$

where $\eta_{\rho\mu}$ is an as-yet-unspecified variable, can be an identity, that is, satisfied for any $\mathcal{L}$, only if

$$\frac{\partial \mathcal{L}}{\partial U_\mu} = \frac{d\eta_{\rho\mu}}{dU_\mu}\frac{\partial \mathcal{L}}{\partial \eta_{\rho\mu}} + \frac{d\eta_{\rho\mu}^*}{dU_\mu}\frac{\partial \mathcal{L}}{\partial \eta_{\rho\mu}^*}$$

This means that

$$\frac{d\eta_{\rho\mu}}{dU_\mu} = ig\psi_\rho \qquad \text{and} \qquad \frac{d\eta_{\rho\mu}^*}{dU_\mu} = -ig\psi_\rho^*$$

It is trivial to integrate the last equations, since U_μ, ψ_ρ, and ψ_ρ^* are independent variables in the function $\mathcal{L}[\psi, U]$. Hence $\eta_{\rho\mu} = c_{\rho\mu} + igU_\mu\psi_\rho$ and $\eta_{\rho\mu}^* = c_{\rho\mu}^* - igU_\mu\psi_\rho^*$, where $c_{\rho\mu}$ and $c_{\rho\mu}^*$ are the integration constants. Because

Condition 2 must hold at all values of U_μ, it holds at $U_\mu = 0$ when $\eta_{\rho\mu} = \psi_{\rho,\mu}$ and $\eta^*_{\rho\mu} = \psi^*_{\rho\mu}$. Then it follows that $c_{\rho\mu} = \psi_{\rho,\mu}$ and $c^*_{\rho\mu} = \psi^*_{\rho,\mu}$, showing that

$$\eta_{\rho\mu} = \psi_{\rho,\mu} + igU_\mu\psi_\rho \qquad \text{and} \qquad \eta^*_{\rho\mu} = \psi^*_{\rho,\mu} - igU_\mu\psi^*_\rho$$

which in turn proves the theorem.

The result (7) can be obtained, of course, by simply requiring that the Lagrangian density $\mathcal{L}[\psi, U]$ be *gauge invariant*. However, from this paper's viewpoint, Condition 2 is easier to justify by physical arguments than the *gauge invariance*. Hence, it is more meaningful to postulate Condition 2 and derive the *gauge invariance* from it than vice versa.

3. A PRESCRIPTION FROM QUANTUM MECHANICS

Having made the functional dependence of $\mathcal{L}[\psi, U]$ on U quite specific, we see that the Euler–Lagrange equations for ψ_ρ become more specific, too. Since $\mathcal{N}$ is not a function of ψ_ρ, we get from

$$\frac{d}{dx_\mu}\left(\frac{\partial\mathcal{T}}{\partial\psi_{\rho,\mu}}\right) - \frac{\partial\mathcal{T}}{\partial\psi_\rho} = 0$$

immediately the results

$$\left(\frac{d}{dx_\mu} - igU_\mu\right)\frac{\partial\mathcal{L}}{\partial\eta_{\rho\mu}} - \frac{\partial\mathcal{L}}{\partial\psi_\rho} = 0, \qquad \left(\frac{d}{dx_\mu} + igU_\mu\right)\frac{\partial\mathcal{L}}{\partial\eta^*_{\rho\mu}} - \frac{\partial\mathcal{L}}{\partial\psi^*_\rho} = 0 \tag{8}$$

where, as before,

$$\eta_{\rho\mu} = \frac{\partial\psi_\rho}{\partial x_\mu} + igU_\mu\psi_\rho, \qquad \eta^*_{\rho\mu} = \frac{\partial\psi^*_\rho}{\partial x_\mu} - igU_\mu\psi^*_\rho$$

This important result deserves to be stated in words:

Given a solitary wave equation derivable from a Lagrangian density which satisfies Conditions 1 and 2, one can find the equation of the same solitary wave when interacting with a given external real 4-potential field U_μ by replacing all space and time derivatives in the given equation according to the following rule:

$$\frac{\partial}{\partial x_\mu} \to \frac{\partial}{\partial x_\mu} + igU_\mu, \qquad \mu = 0, 1, 2, 3 \tag{9}$$

In this, one of course recognizes the *fundamental prescription of quantum mechanics*

$$i\hbar\frac{\partial}{\partial x_\mu} \to i\hbar\frac{\partial}{\partial x_\mu} + eA_\mu \tag{10}$$

for the energy and momentum operators in the presence of an interaction. But unlike the QM prescription, which is intended to be used only with the equations of quantum mechanics, all of which are *linear*, the rule (9), as shown already, can be used with a much larger class of equations—*both linear and nonlinear.*

4. THE LAGRANGIAN FUNCTION FOR AN INTERACTING SOLITARY WAVE

All ψ-field/U-field systems may naturally be divided into two classes: those with slowly varying and those with rapidly varying U-fields, both in space or in time. A criterion to distinguish between the two classes will be derived at the end of this section. Consider the first case, when the U-field varies sufficiently slowly so that it can be treated as a constant in the SW Lagrangian density (but not in the SW Lagrangian function, of course). Then one can draw a few more conclusions without any additional assumptions. That is, one can find the *Lagrangian function*

$$L(U) = \int \mathcal{L}[\psi, U]\, d^3x \tag{11}$$

which governs the macroscopic motion of the solitary wave. Here the integral domain is all space and the integral is nonzero and finite by continuity considerations and by the SW definition. At this level the integral obviously cannot be calculated directly. However, we do know that if the density $\mathcal{L}[\psi, U]$ was given explicitly, the result of the integration would be a function of U_μ, evaluated at the SW position, and of the SW velocity. Therefore, we need to specify the position of the solitary wave as a whole. It is to be given by the k-coordinates of its *center of charge* X_k, defined as follows:

$$X_k = \frac{1}{Q} \int x_k \mathcal{D}_0\, d^3x \qquad \text{for} \quad k = 1, 2, 3 \tag{12}$$

where

$$\mathcal{D}_0 = i\psi_\rho \frac{\partial \mathcal{L}}{\partial \psi_{\rho,0}} - i\psi_\rho^* \frac{\partial \mathcal{L}}{\partial \psi_{\rho,0}^*}, \qquad Q = \int \mathcal{D}_0\, d^3x$$

are the *SW charge density* and the *SW total charge*, respectively. Since the U_μ are assumed to be constant within the space occupied by the SW, the integral (11) can be differentiated with respect to U_μ:

$$\frac{\partial L}{\partial U_\mu} = \int \frac{\partial}{\partial U_\mu} \mathcal{L}[\psi, U]\, d^3x = g \int \mathcal{D}_\mu\, d^3x$$

Here, the second equality is just an application of Condition 2—equation (6). This gives for $\mu = 0$ and $\mu = k$, respectively,

$$\frac{\partial L}{\partial U_0} = g \int \mathfrak{D}_0 \, d^3x = gQ \tag{13}$$

$$\frac{\partial L}{\partial U_k} = g \int \mathfrak{D}_k \, d^3x \tag{14}$$

with $\mathfrak{D}_k$ being the *current density* associated with ψ_p. Using the conservation law for $\mathfrak{D}_\mu$ and the definition of the *SW center* X_k, one can express the k-component of the SW *velocity* V_k as

$$V_k = \frac{dX_k}{dt} = \frac{1}{Q} \int \frac{\partial \mathfrak{D}_0}{\partial t} x_k \, d^3x = -\frac{1}{Q} \sum_{i=1}^{3} \int \frac{\partial \mathfrak{D}_i}{\partial x_i} x_k \, d^3x = \frac{1}{Q} \int \mathfrak{D}_k \, d^3x$$

The last result and (14) produce

$$\frac{\partial L}{\partial U_k} = gQV_k \qquad \text{for} \quad k = 1, 2, 3 \tag{15}$$

Thus, we obtain a system of four partial differential equations: equations (13) and (15). Since the total charge Q is conserved, it cannot depend on U_μ, which makes the solution of the above system obvious:

$$L(U) = gQ(U_0 + U_1V_1 + U_2V_2 + U_3V_3) + L(0) = gQ(U_0 + \overline{U} \cdot \overline{V}) + L(0) \tag{16}$$

One can verify it by substituting it in (13) and (15). Here, $L(0)$ is the integration constant, which equals the space integral of the free ψ-field Lagrangian density

$$L(0) = \int \mathcal{L}[\psi, 0] \, d^3x. \tag{17}$$

This integral is finite and nonzero by definition—the SW definition.

The criterion for the U-field maximum rate of change [at which equation (16) still holds] is derived as follows. Let $U_{\max}$ and $U_{\min}$ be, respectively, the maximal and minimal U-field values within the SW extent at a given moment in a given reference frame. Denote by $L(U_{\max})$ and $L(U_{\min})$ the Lagrangian function (11) when the integral is calculated with the corresponding fixed values of U. (Naturally, we assume that U is a monotonic function of the coordinates within the SW extent.) Then if the inequality

$$L(U_{\max}) - L(U_{\min}) << L(U_{\min})$$

holds, one is permitted to treat U as a constant when integrating (11). With the use of previous results we can write

$$L(U_{\max}) - L(U_{\min}) \approx \left. \frac{\partial L(U)}{\partial U_\mu} \right|_{U_{\min}} \delta U_\mu = gQ(\delta U_0 + \overline{V} \cdot \delta \overline{U})$$

Hence, the criterion is

$$|\delta U_0| + |\overline{V}| \cdot |\delta \overline{U}| << \left| \frac{L(U)}{gQ} \right| \tag{18}$$

where $|\delta U_0|$ and $|\delta \overline{U}|$ are the *maximal variations* of the U-field components over the solitary wave extent and $|\overline{V}|$ is the SW velocity magnitude. For an estimate of the SW radius R one can use

$$R^2 = \frac{1}{Q} \sum_{k=1}^{3} \int \mathcal{D}_0 (x_k - X_k)^2 \, d^3x$$

where the integral domain is all space.

5. CONCLUSIONS

(a) The first term in the SW Lagrangian function (16) is precisely the same as the interaction term in the Lagrangian function for a point charge with magnitude $q = gQ$ moving in an electromagnetic 4-potential field $A_\mu = U_\mu$. For the standard treatment of point-charge Lagrangian functions the reader is referred to Goldstein (1981, Sections 7.8 and 8.1) or Jackson (1975, Section 12.1). If we choose the density $\mathcal{L}[\psi, 0]$ to be Lorentz invariant, then the integral (17) gives

$$L(0) = M\sqrt{1 - V^2} \qquad (c = 1)$$

where V is the magnitude of the SW velocity and M is a constant determined only by the properties of the specific SW equation, which of course should be identified with the *solitary wave rest mass*. Then the expression (16) is completely identical with the Lagrangian function for a relativistic charge. For small velocities, $L(0) = M - \frac{1}{2}MV^2$, giving the nonrelativistic case, after the constant term M is discarded. The following conclusion will be stated as a theorem:

Theorem 2. If the equations which possess a solitary wave solution are derivable from a Lorentz-invariant Lagrangian density satisfying Conditions 1 and 2, and if the U-field is slowly varying, according to criterion (18), then the equations for the macroscopic motion of that solitary wave *are identical* with those describing the motion of a point charge proportional to Q in an electromagnetic 4-potential field proportional to U. This identity holds *regardless* of the specific ψ-field equations.

Proof. It was already shown that the SW Lagrangian function obtained from any SW Lagrangian density (satisfying the above conditions) is identical, within constants of proportionality, to that of a point charge in an electromagnetic field. Since the equations of motion are uniquely determined by the corresponding Lagrangian functions, the theorem follows.

(b) The motion of a solitary wave in a field with a scalar potential W (of unspecified physical nature—not only electrical) is a special case of (a). This case is obtained by setting

$$U_0 = \frac{W}{gQ} \qquad \text{and} \qquad U_k \equiv 0, \quad k = 1, 2, 3$$

in (16). Then the SW Lagrangian function

$$L = W + L(0)$$

corresponds to that of a point mass with a potential energy W. The ψ-field equation(s) obtained with the use of the single substitution

$$\frac{\partial}{\partial x_0} \rightarrow \frac{\partial}{\partial x_0} + i\frac{W}{Q} \tag{19}$$

then are analogous to the Schrödinger equation describing the motion of a particle in a potential W. The substitution (19) is analogous, of course, to the QM prescription for the energy operator.

(c) It is very satisfying that, starting with equations for the SW ψ-field which must be *nonlinear* (in order to possess solitary wave solutions), one arrives at equations for the SW macroscopic motion which are *strictly linear*, both in $\partial U_\mu/\partial x_\nu$ and in gQ.

(d) The reader most probably has noticed that conclusion (a) is analogous to the Ehrenfests theorem in quantum mechanics. There is a very good reading on this theorem in Kramer (1958, Section 30).

(e) The behavior of a solitary wave will differ drastically from that of a point charge or of a point mass if the U-field rate of change is comparable to that of the SW ψ-field. This is because when the criterion (18) is violated, one cannot differentiate with respect to U_μ under the integral sign in (11), and consequently the SW Lagrangian function, if it can be defined at all, will differ drastically from that of a point charge or a point mass. Analogies with quantum mechanics come to mind again; however, their discussion is beyond the scope of the present topic.

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